# Operations for D-Algebraic Functions

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# Some references


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# Outline

# 1. Introduction

# D-Algebraic Functions

## What it means to be D-algebraic

A function $f := f(x)$ is said to be differentially algebraic (or D-algebraic) if there exist $n \in \mathbb{N}$, and a polynomial $P \in \mathbb{K}[x, y_0, \ldots, y_n]$ such that

$$P\left(x, y_0 = f(x), \ldots, y_n = f^{(n)}(x)\right) = 0. \tag{1}$$

The differential equation resulting from (1) is called Algebraic Differential Equation (ADE).

We focus on operations with (arbitrary) solutions to ADEs. See [van Der Hoeven, 2019] for initial conditions.

# D-Algebraic Functions

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## Example

$\log(\sin)$, $\frac{\cos}{\sin}$, and $(\log(\sin))''$ are D-algebraic and satisfy the ADEs

- $f''(x) + (f'(x))^2 + 1 = 0$,
- $f'(x) + (f(x))^2 + 1 = 0$, and
- $(f'(x))^2 + 4\,(f(x))^3 + 4\,(f(x))^2 = 0$, respectively.

# Operations

## Theorem

*The set $\mathcal{A}$ of D-algebraic functions is a field. Moreover, $\mathcal{A}$ is closed under composition, taking functional inverse, and taking derivatives.*

Proof: See [Ait El Manssour, Sattelberger, and T., 2023] and [van Der Hoeven, 2019, Bernardi, O., Bousquet-Mélou, M., and Raschel, K., 2020].

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## Operations

Let $f$ and $g$ be D-algebraic functions.

- Unary operations (only one operand from $\mathcal{A}$): $\alpha(f)$, $\alpha \in \{(\cdot)^n; (\cdot)^{(n)}; (\cdot)^{-1}; R(y)|_{y=\cdot}, R \in \mathbb{K}(x, y)\}$.
- Binary operations: $f \alpha g$, with $\alpha \in \{\times, /, +, -, \circ\}$.
- $N$-ary operations: $\alpha(f_1, \ldots, f_N)$, with $\alpha = R(y_1, \ldots, y_N)|_{y_1=f_1,\ldots,y_N=f_N \cdot}, R \in \mathbb{K}(x, y_1, \ldots, y_N)$

# Implementation: NLDE

## The NLDE Package

1. **N**on**L**inear algebra and **D**ifferential **E**quations (or **N**on**L**inear **D**ifferential **E**quations).
2. Implemented in Maple language for computing ADEs.
3. Gröbner bases elimination: `PolynomialIdeal` and `Groebner`.
4. MathRepo webpage:
   **https://mathrepo.mis.mpg.de/OperationsForDAlgebraicFunctions**
5. Github page (everything (including the source)):
   **https://github.com/T3gu1a/D-algebraic-functions**.
6. Loading:

```
> restart;

> libname:=currentdir(), libname:

> with(NLDE)
```

# Differential Elimination

**Existing methods**: Rosendfeld-Gröbner algorithm and Thomas decomposition.

# Differential Elimination

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## [T., 2023, Example 5]

- Example: $P_1 := {y_1'}^2 + y_1^2 - 1,\, P_2 := y_2' - y_2,\, P_3 := {y_3'}^3 + {y_3'}^2 + 3$
- Find a third-order differential polynomial $T$: $T\left(z(x) = \frac{f_1(x)\, f_3(x)}{f_2(x)}\right)$, where $P_1(f_1) = P_2(f_2) = P_3(f_3) = 0$.
- We find

$$T := \left(z'' + 2z' + z\right)\left(z'' + 2z' + 2z\right)^2 \left(12z^2 + 32z'z + 20z''z + 6z^{(3)}z + 24{z'}^2 + 30z''z' + 10z^{(3)}z' + 9{z''}^2 + 6z^{(3)}z'' + {z^{(3)}}^2\right) \quad (2)$$

The method used is inspired by the work in [Hong, Ovchinnikov, Pogudin, and Yap, 2020].

# 2. The NLDE Package: Some Features

# Univariate Arithmetic

## Example (Rational expressions in $\mathcal{A}$)

```
> ADE1:=diff(y1(x),x)^3+y1(x)+1=0:
> ADE2:=diff(y2(x),x)^2-y2(x)-1=0:
> NLDE:-arithmeticDalg([ADE1,ADE2],[y1(x),y2(x)],z=y1+y2)
```

$$-24\left(\frac{d^2}{dx^2}z(x)\right)^3+36\left(\frac{d^2}{dx^2}z(x)\right)^2-18\frac{d^2}{dx^2}z(x)+8\frac{d^3}{dx^3}z(x)+3=0 \tag{3}$$

```
> Res:=NLDE:-arithmeticDalg([ADE1,ADE2],[y1(x),y2(x)],z=y1+y2,lho=false):
```

$$\begin{aligned}&\left(\left(\frac{d}{dx}z(x)\right)^3+z(x)+2\right)\left(\left(\frac{d}{dx}z(x)\right)^2-z(x)-2\right)\left(216\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)^3-216z(x)\left(\frac{d^2}{dx^2}z(x)\right)^3\right.\\&-324\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)^2-432\left(\frac{d^2}{dx^2}z(x)\right)^3+144\left(\frac{d}{dx}z(x)\right)\left(\frac{d^2}{dx^2}z(x)\right)^2+324z(x)\left(\frac{d^2}{dx^2}z(x)\right)^2\\&+162\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)+648\left(\frac{d^2}{dx^2}z(x)\right)^2-144\left(\frac{d}{dx}z(x)\right)\left(\frac{d^2}{dx^2}z(x)\right)-162z(x)\left(\frac{d^2}{dx^2}z(x)\right)\\&\left.-27\left(\frac{d}{dx}z(x)\right)^2-300\frac{d^2}{dx^2}z(x)+36\frac{d}{dx}z(x)+27z(x)+50\right)=0\end{aligned} \tag{4}$$

# Composition

## Example (The exponential of the Painlevé transcendent of type I.)

```
> ADE1:= diff(y1(x), x) - y1(x) = 0:
> ADE2:= diff(y2(x),x,x)=6*y2(x)^2+x: # The transcendent
> NLDE:-composeDalg([ADE1,ADE2],[y1(x),y2(x)],z(x))
```

$$
\begin{aligned}
&24x\left(\frac{d}{dx}z(x)\right)^2 z(x)^4 + z(x)^6 - 2z(x)^5\left(\frac{d^3}{dx^3}z(x)\right) + 6\left(\frac{d^2}{dx^2}z(x)\right)\left(\frac{d}{dx}z(x)\right)z(x)^4 \\
&+\left(\frac{d^3}{dx^3}z(x)\right)^2 z(x)^4 - 4\left(\frac{d}{dx}z(x)\right)^3 z(x)^3 - 24\left(\frac{d^2}{dx^2}z(x)\right)\left(\frac{d}{dx}z(x)\right)^2 z(x)^3 \\
&-6\left(\frac{d^3}{dx^3}z(x)\right)\left(\frac{d^2}{dx^2}z(x)\right)\left(\frac{d}{dx}z(x)\right)z(x)^3 + 24\left(\frac{d}{dx}z(x)\right)^4 z(x)^2 \\
&+4\left(\frac{d^3}{dx^3}z(x)\right)\left(\frac{d}{dx}z(x)\right)^3 z(x)^2 + 9z(x)^2\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)^2 \\
&-12z(x)\left(\frac{d}{dx}z(x)\right)^4\left(\frac{d^2}{dx^2}z(x)\right) + 4\left(\frac{d}{dx}z(x)\right)^6 = 0
\end{aligned}
\tag{5}
$$

# Functional inverse

## Example (Weierstrass $\wp$ and Painlevé Transcendent I)

- ```
  > ADE:=diff(y1(x),x)^2=4*y1(x)^3-g2*y1(x)-g3: # Weierstrass
  > NLDE:-invDalg(ADE,y1(x),z(x))
  ```

$$1 + \left(-4x^3 + g2x + g3\right)\left(\frac{d}{dx}z(x)\right)^2 = 0. \tag{6}$$

- ```
  > ADE:=diff(y(x),x,x)=6*y(x)^2+x: # The transcendent
  > NLDE:-invDalg(ADE,y(x),z(x))
  ```

$$-6x^2\left(\frac{d}{dx}z(x)\right)^3 - z(x)\left(\frac{d}{dx}z(x)\right)^3 - \frac{d^2}{dx^2}z(x) = 0 \tag{7}$$

# Differentiation

## Example (Derivatives of the Weierstrass ℘ function)

- 
```
> ADE:=diff(y1(x),x)^2=4*y1(x)^3-g2*y1(x)-g3:
> NLDE:-diffDalg(ADE,y1(x))
```

$$-1728\mathrm{y1}(x)^4+64g2^3-192g2\left(\frac{d}{dx}\mathrm{y1}(x)\right)^2-3456g3\mathrm{y1}(x)^2+128\left(\frac{d}{dx}\mathrm{y1}(x)\right)^3-1728g3^2=0. \tag{8}$$

- 
```
> NLDE:-diffDalg(ADE,y1(x),2)
```

$$\begin{aligned}&16g2^5+64g2^4\mathrm{y1}(x)+16g2^3\mathrm{y1}(x)^2-160g2^2\mathrm{y1}(x)^3-64g2\mathrm{y1}(x)^4\\&+128\mathrm{y1}(x)^5-432g2^2g3^2-1728g2\,g3^2\mathrm{y1}(x)-72g2g3\left(\frac{d}{dx}\mathrm{y1}(x)\right)^2\\&-1728g3^2\mathrm{y1}(x)^2-144g3\mathrm{y1}(x)\left(\frac{d}{dx}\mathrm{y1}(x)\right)^2-3\left(\frac{d}{dx}\mathrm{y1}(x)\right)^4=0.\end{aligned} \tag{9}$$

# DDfinite to D-Algebraic

Jiménez-Pastor, A., Pillwein, V., Singer, M., 2020. Some structural results on $D^n$-finite functions. Advances in Applied Mathematics 117, ID 102027 (29 pages).

## Example (ADE for Mathieu functions)

```
> cosDE:=DEtools:-FindODE(cos(2*x),C(x)):
> ADE:=diff(y(x),x,x)+(a-2*q*C)*y(x)=0:
> NLDE:-DDfiniteToDalg(ADE,y(x),[cosDE],[C(x)])
```

$$4ay(x)^3 + 4y(x)^2\left(\frac{d^2}{dx^2}y(x)\right) + y(x)^2\left(\frac{d^4}{dx^4}y(x)\right) - 2\left(\frac{d^3}{dx^3}y(x)\right)y(x)\left(\frac{d}{dx}y(x)\right) - \left(\frac{d^2}{dx^2}y(x)\right)^2 y(x) + 2\left(\frac{d}{dx}y(x)\right)^2\left(\frac{d^2}{dx^2}y(x)\right) = 0. \tag{10}$$

# Take away

## NLDE can

- compute ADEs satisfied by rational expressions of univariate D-algebraic functions (`arithmeticDalg`);
- compute ADEs satisfied by compositions of D-algebraic functions (`composeDalg`);
- compute ADEs satisfied by functional inverses of D-algebraic functions (`invDalg`);
- compute ADEs satisfied by derivatives of D-algebraic functions (`diffDalg`);
- compute ADEs satisfied by DD-finite functions (`DDfiniteToDalg`);
- "compute" ADEs satisfied by rational expressions of multivariate D-algebraic functions (`MDalg:-arithmeticMDalg`);
- will try to improve `AnsatzDalg` for D-algebraic functions satisfying higher order ADEs.

If $\wp(x)$ satisfies

$$y'(x)^2 = 4y(x)^3 - g_2\, y(x) - g_3.$$

Then the functional inverse $\wp^{-1}$ of $\wp$ satisfies

$$1 + \left(-4x^3 + g2x + g3\right) y'(x)^2 = 0.$$

Computed with **`NLDE:-invDalg`**

Download **`NLDE`** at **`https://mathrepo.mis.mpg.de/OperationsForDAlgebraicFunctions`**

# *Thank You!*