Differentially Algebraic Functions

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July 14, 2023

Some references

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  D-algebraic functions.
  arXiv preprint arXiv:2301.02512.
- Bernardi, O., Bousquet-Mélou, M., and Raschel, K. (2020).
  Counting quadrant walks via Tutte's invariant method.
  Discrete Mathematics & Theoretical Computer Science.
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  Global identifiability of differential models.
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  Arithmetic of D-algebraic functions.
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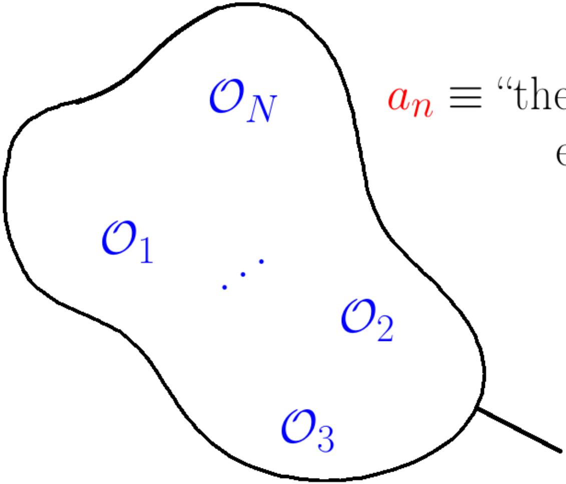
  Computing with D-algebraic power series.
  Applicable Algebra in Engineering, Communication and Computing, 30(1):17–49.

Outline

1. Introduction

Motivation

Ordinary generating functions may be adapted to finding moments of distributions.



$a_n \equiv$ "the number of objects in $\mathcal{S}$ with exactly n properties"

$$\mathcal{S} = \{\mathcal{O}_1, \ldots, \mathcal{O}_N\}$$

Generating Functions

- The average number of properties that an object in $\mathcal{S}$ has is $\mu = \frac{1}{N}\sum_{n=1}^{\infty} na_n$
- The standard deviation is given by $\sigma = \sqrt{\frac{1}{N}\sum_{n=0}^{\infty}(n-\mu)^2 a_n}$

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Applications of generating functions (H. S. Wilf, Generatingfunctionology, 1990)

- $\mu = \left.\frac{f'(x)}{f(x)}\right|_{x=1}$
- $\sigma = \left.\sqrt{(\log(f(x)))' + (\log(f(x)))''}\right|_{x=1}$

D-Finite Functions

What is meant by knowing a function?

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Holonomic or D-finite representation

$$c_d(x)\, f^{(d)}(x) + \cdots + c_0(x)\, f^{(0)}(x) = 0, \quad d \in \mathbb{N}, c_d, \ldots, c_0 \in \mathbb{K}[x], c_d \neq 0 \\ f(x_0), \ldots, f^{(d-1)}(x_0), x_0 \in \mathbb{K}. \tag{1}$$

In higher dimension, one studies holonomic systems with *D*-modules techniques (Satterlberger and Sturmfels, 2019; Koutschan Ph.D. thesis, 2009).

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Example

- The Catalan number C_n: $C_3 = 5$: ((())) ()(()) ()()() (())() (()())
  $\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n$. One can deduce C_n with **FPS**.
- Manuel Kauers and Peter Paule, The Concrete Tetrahedron, 2011.

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- Manuel Kauers and Peter Paule, The Concrete Tetrahedron, 2011.
- If $f := \sin$, then $\log(f)$, $\frac{f'}{f}$ and $(\log(f))''$ are not D-finite.

D-Algebraic Functions

"Natural definition"

A function $f := f(x)$ is said to be differentially algebraic (or D-algebraic) if there exist $n \in \mathbb{N}$, and a polynomial $P \in \mathbb{K}[x, y_0, \ldots, y_n]$ such that

$$P\left(x, y_0 = f(x), \ldots, y_n = f^{(n)}(x)\right) = 0. \tag{2}$$

The differential equation resulting from (2) is called Algebraic Differential Equation (ADE).

We focus on operations with (arbitrary) solutions to ADEs. See [van Der Hoeven, 2019] for initial conditions.

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Example

$\log(\sin)$, $\frac{\cos}{\sin}$, and $(\log(\sin))''$ are D-algebraic and satisfy the ADEs

- $f''(x) + (f'(x))^2 + 1 = 0$,
- $f'(x) + (f(x))^2 + 1 = 0$, and
- $(f'(x))^2 + 4\,(f(x))^3 + 4\,(f(x))^2 = 0$, respectively.

Operations for D-Algebraic Functions

Theorem

The set $\mathcal{A}$ of D-algebraic functions is a field. Moreover, $\mathcal{A}$ is closed under composition, taking functional inverse, and taking derivatives.

Details proofs can be found in [Ait El Manssour, Sattelberger, and T., 2023]. Other references include [van Der Hoeven, 2019, Bernardi et al., 2020].

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Operations

Let f and g be D-algebraic functions.

- Binary operations:
  $f \alpha g$, with $\alpha \in \{\times, /, +, -, \circ\}$.
- Unary operations (only one operand from $\mathcal{A}$):
  $\alpha(f)$, $\alpha \in \{(\cdot)^n; (\cdot)^{(n)}; (\cdot)^{-1}; R(y)|_{y=\cdot}, R \in \mathbb{K}(x, y)\}$.

MathRepo webpage:
https://mathrepo.mis.mpg.de/DAlgebraicFunctions/
Software presentation of the Maple implementation at **ISSAC'23**.

2. Arithmetic Operations

Some Differential Algebra

- Let $R := (\mathbb{K}[x], \frac{d}{dx})$ be a differential ring, $\mathbb{K} \supset \mathbb{Q}$.
- We call $S_y := R[y, y', \ldots, y^{(n)}, \ldots]$, $y^{(n)} := \frac{d^n}{dx^n} y$, a "univariate" differential polynomial ring. The variable y and all its derivatives are seen as "one differential variable" called *differential indeterminate*.
- A differential ideal I, is an ideal $I \in S_y$, that is closed under $\frac{d}{dx}$. For $p_1, \ldots, p_k \in S_{y_1,\ldots,y_l}$ (multivariate), $l, k \in \mathbb{N}$, we denote by $\langle p_1, \ldots, p_k \rangle^{(\infty)}$, the minimal differential ideal containing $p_1, \ldots, p_k$ and their derivatives.

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Definition (D-algebraic function)

A function $f := f(x)$ is D-algebraic with respect to x, say $f \in \mathcal{A}_x$, if there exists $P \in S_y$, such that $P(y = f(x)) = 0$.

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For convenience, we use different differential indeterminates for distinct D-algebraic functions:

$$\text{For } f, g \in \mathcal{A}_x, \text{ we consider } P \in S_{y_1}, Q \in S_{y_2}, P(y_1 = f(x)) = Q(y_2 = g(x)) = 0.$$

Note: $P, Q \in S_{y_1,y_2}$.

Differential elimination

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- We work on $S_{y_1,y_2,z}$. Let r be the numerator of $z - y_1 \alpha y_2$, and consider the differential ideal

$$I_{P,Q,r} := \langle P, Q, r \rangle^{(\infty)} \subset S_{y_1,y_2,z}. \tag{3}$$

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Theorem (Ritt-Raudenbash)

Let I be a differential ideal in S_y. The radical ideal $\sqrt{I}$ is a differential ideal, and moreover, there exist prime differential ideals $I_1, \ldots, I_m$ such that

$$\sqrt{I} = I_1 \cap I_2 \cap \cdots \cap I_m \tag{4}$$

Method I

We want to find a differential polynomial

$$T \in I_{P,Q,r} \cap S_z$$

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Method I

1. For $N \in \mathbb{N}$, let $I_{P,Q,r}^{(\leq N)}$ be the algebraic ideal defined by P, Q, r and their first N derivatives, and $S_z^{(\leq N)} := \mathbb{K}[x][z, z' \ldots, z^{(N)}]$ (a polynomial ring).

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2. Use Gröbner bases to iterate the computations

$$N \longleftarrow N + 1; \tag{5}$$

$$J \longleftarrow I_{P,Q,r}^{(\leq N)} \cap S_z^{(\leq N)}. \tag{6}$$

3. Until $J \neq \{0\}$.
4. Return $T \in J$, such that T has minimum degree among the generators of the minimal order in J.

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Questions: how many times do we need to iterate? What can we say about the order?

Dynamical system

Consider $S_{\mathbf{y},z}$, where $\mathbf{y} = (y_1, \ldots, y_n)$, and the dynamical system

$$\begin{cases} \mathbf{y}' = \mathbf{A}(\mathbf{y}) \\ z = B(\mathbf{y}) \end{cases}, \qquad (\mathcal{M})$$

where $\mathbf{A} = (A_1, \ldots, A_n) \in \mathbb{K}(x)(y_1, \ldots, y_n)^n$, $B \in \mathbb{K}(x)(y_1, \ldots, y_n)$.

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Differential ideal associated to $(\mathcal{M})$

Let $Q := \mathrm{lcm}(\text{denominators in } \mathcal{M})$, $A_i = a_i/Q$, $i = 1, \dots, n$, $B = b/Q$, and

$$I_{\mathcal{M}} := \langle Q\,\mathbf{y}' - \mathbf{a}(y),\, Q\,z - b(\mathbf{y})\rangle : Q^{\infty} \subset S_{\mathbf{y},z}, \tag{7}$$

$$I_{\mathcal{M}}^{(\leq j)} := \langle (Q\,\mathbf{y}' - \mathbf{a}(y))^{(<j)}, (Q\,z - b(\mathbf{y}))^{\leq j}\rangle \subset S_{\mathbf{y},z}^{(\leq j)} \tag{8}$$

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[Hong, Ovchinnikov, Pogudin, and Yap, 2020]

$$I_{\mathcal{M}}^{(\leq n)} \cap S_z^{(\leq n)} \neq \{0\} \tag{9}$$

Method II

Computing ADEs from dynamical systems

1. Let
$$P := P(y_1, \ldots, y_1^{(n_1)}), \quad Q := Q(y_2, \ldots, y_2^{(n_2)}), \tag{10}$$
and assume $\deg_{y_1^{(n_1)}}(P) = \deg_{y_2^{(n_2)}}(Q) = 1$.

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and assume $\deg_{y_1^{(n_1)}}(P) = \deg_{y_2^{(n_2)}}(Q) = 1$.
2. Consider the differential indeterminates $w_1, \ldots, w_{n_1+n_2}$, such that
$$w_1 := y_1, \ldots, w_{n_1} := y_1^{(n_1-1)}, w_{n_1+1} := y_2, \ldots, w_{n_1+n_2} := y_2^{(n_2-1)} \tag{11}$$

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3. Deduce the ADE sought from the dynamical system
$$\begin{cases} w_1' = w_2 \\ \dots \\ w_{n_1}' = r_P(w_1, \dots, w_{n_1-1}), \quad r_P := \text{Solve}(P, y_1^{(n_1)}) \\ \dots \\ w_{n_1+n_2}' = r_Q(w_{n_1+1}, \dots, w_{n_1+n_2-1}), \quad r_Q := \text{Solve}(Q, y_2^{(n_2)}) \\ z = w_1 \alpha w_{n_1+1} \end{cases} . \tag{12}$$

Examples

- $f(x) := \exp(x)$, $P := y_1' - y_1$; and $g(x) := \log(x)$, $Q(x) := x\, y_2' - 1$. Find T such that $T\left(z(x) = \frac{f(x)}{g(x)}\right) = 0$.
- We find

$$T := x\, z'' \, z - 2\, x\, {z'}^{2} + (2\, x + 1)\, z'\, z - (x + 1)\, z^2$$

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$$T := x\, z''\, z - 2\, x\, z'^2 + (2\, x + 1)\, z'\, z - (x + 1)\, z^2$$
- $P := y_1'' + y_1'^2$ and $Q := y_2' - y_2^2$. Find T: $T\,(z(x) = f(x) + g(x)) = 0$, where $P(f) = Q(g) = 0$.
- We find
$$\begin{aligned} T := {} & 336\, z'\, z''^2\, {z^{(3)}}^2 + 1024\, z'^9 + 432\, z'^8 - 96\, z'^3\, z''\, {z^{(3)}}^2 \\ & - 384\, z'\, z''\, {z^{(3)}}^2 - 576\, z'^2\, z''^2\, z^{(3)} + 768\, z'\, z''^3\, z^{(3)} + 672\, z'^2\, z''^3\, z^{(3)} \\ & - 1536\, z'^3\, z''^2\, z^{(3)} - 1152\, z'^5\, z''\, z^{(3)} - 32\, {z^{(3)}}^3 - 16\, z''^6 - 288\, z''^5 \\ & + 288\, z''^3\, z^{(3)} + 24\, z''^2\, {z^{(3)}}^2 - 1008\, z'^2\, z''\, {z^{(3)}}^2 + 432\, z''^4 + 24\, z''\, {z^{(3)}}^3 \\ & - 768\, z'^6\, z''^2 + 4608\, z'^3\, z''^3 + 4224\, z'^7\, z'' + 1728\, z'^6\, z'' + 6336\, z'^5\, z''^2 \\ & - 672\, z'^4\, z''^3 + 192\, z'^3\, z''^4 + 2592\, z'^4\, z''^2 - 456\, z'^4\, {z^{(3)}}^2 + 576\, z'^2\, z''^4 \\ & - 384\, z'\, z''^5 - 192\, z'^3\, {z^{(3)}}^2 + 1728\, z'^2\, z''^3 - 48\, z'^2\, {z^{(3)}}^3 + 1728\, z''^4\, z' \\ & - 192\, z''^4\, z^{(3)} - 8\, z''^3\, {z^{(3)}}^2 - 96\, z'\, {z^{(3)}}^3 - {z^{(3)}}^4 - 480\, z'^4\, z''\, z^{(3)}. \end{aligned} \tag{13}$$

Related work

- Jiménez-Pastor, A., Pillwein, V., Singer, M., 2020. Some structural results on D^n-finite functions. Advances in Applied Mathematics 117, ID 102027 (29 pages).

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- Robertz, D., 2014. Thomas Decomposition of Differential Systems. In Robertz, D., 2014. Formal Algorithmic Elimination for PDEs. Springer 2121. Switzerland. Section 2.2.

Related work

Improved Method II [T., 2023]

- Example: $P_1 := {y_1'}^2 + y_1^2 - 1, P_2 := y_2' - y_2, P_3 := {y_3'}^3 + {y_3'}^2 + 3$
- Find a third-order differential polynomial T: $T\left(z(x) = \frac{f_1(x)\, f_3(x)}{f_2(x)}\right)$, where $P_1(f_1) = P_2(f_2) = P_3(f_3) = 0$.

Related work

Improved Method II [T., 2023]

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- We find

$$T := \left(z'' + 2z' + z\right)\left(z'' + 2z' + 2z\right)^2 \left(12z^2 + 32z'z + 20z''z + 6z^{(3)}z + 24{z'}^2 + 30z''z' + 10z^{(3)}z' + 9{z''}^2 + 6z^{(3)}z'' + {z^{(3)}}^2\right) \tag{14}$$

These non-"l.h.o." cases are often well handled by this approach.

3. More Operations and Implementation

Composition

The main idea

- Let $f, g \in \mathcal{A}_x$, $P := P(y_1, \ldots, y_1^{(n_1)})$, $Q := Q(y_2, \ldots, y_2^{(n_2)})$, $P(f) = Q(g) = 0$.
- How to find $T \in S_z$, $T\,(z = f \circ g) = 0$?

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- Observation:

$$\begin{aligned}
(f(g))' &= g'\, f'(g)\,, \\
(f(g))'' &= g''\, f'(g) + g'^2\, f''(g)\,, \\
(f(g))^{(3)} &= g^{(3)}\, f'(g) + g''\, g'\, f''(g) + g'^3\, f^{(3)}(g) \\
\ldots
\end{aligned} \tag{15}$$

Hence a linear system in y_1 with coefficients in terms of z and y_2.

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  Hence a linear system in y_1 with coefficients in terms of z and y_2.
- Elimination of y_1 by linear algebra.
- Proceed by constructing a dynamical system of dimension $n_1 + n_2$, and deduce the ADE sought.

Implementation in Macaulay2 and Maple

- **https://mathrepo.mis.mpg.de/DAlgebraicFunctions/**
- **https://mathrepo.mis.mpg.de/OperationsForDAlgebraicFunctions/**

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The NLDE Package

Non**L**inear algebra and **D**ifferential **E**quations (or **N**on**L**inear **D**ifferential **E**quations).

1. Univariate arithmetic: `NLDE:-arithmeticDalg`, `NLDE:-unaryDalg`;
2. Multivariate arithmetic: `NLDE:-MDalg`: `MDalg:-arithmeticMDalg`;
3. Composition: `NLDE:-composeDalg`;
4. Functional inverse: `NLDE:-invDalg`;
5. Differentiation: `NLDE:-diffDalg`;
6. Ansatz approach for univariate arithmetic (δ_k-finite functions): `NLDE:-AnsatzDalg`: `AnsatzDalg:-arithmeticDeltak`, `AnsatzDalg:-unaryDeltak`.

The NLDE Package

Example (Arithmetic)

```
> ADE1:=diff(y1(x),x)^3+y1(x)+1=0:
> ADE2:=diff(y2(x),x)^2-y2(x)-1=0:
> NLDE:-arithmeticDalg([ADE1,ADE2],[y1(x),y2(x)],z=y1+y2)
```

$$-24\left(\frac{d^2}{dx^2}z(x)\right)^3+36\left(\frac{d^2}{dx^2}z(x)\right)^2-18\frac{d^2}{dx^2}z(x)+8\frac{d^3}{dx^3}z(x)+3=0 \quad (16)$$

```
> Res:=NLDE:-arithmeticDalg([ADE1,ADE2],[y1(x),y2(x)],z=y1+y2,lho=false):
```

$$\begin{aligned}&\left(\left(\frac{d}{dx}z(x)\right)^3+z(x)+2\right)\left(\left(\frac{d}{dx}z(x)\right)^2-z(x)-2\right)\left(216\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)^3-216z(x)\left(\frac{d^2}{dx^2}z(x)\right)^3\right.\\&-324\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)^2-432\left(\frac{d^2}{dx^2}z(x)\right)^3+144\left(\frac{d}{dx}z(x)\right)\left(\frac{d^2}{dx^2}z(x)\right)^2+324z(x)\left(\frac{d^2}{dx^2}z(x)\right)^2\\&+162\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)+648\left(\frac{d^2}{dx^2}z(x)\right)^2-144\left(\frac{d}{dx}z(x)\right)\left(\frac{d^2}{dx^2}z(x)\right)-162z(x)\left(\frac{d^2}{dx^2}z(x)\right)\\&\left.-27\left(\frac{d}{dx}z(x)\right)^2-300\frac{d^2}{dx^2}z(x)+36\frac{d}{dx}z(x)+27z(x)+50\right)=0\end{aligned} \quad (17)$$

The NLDE Package

Example (Composition)

The exponential of the Painlevé transcendent of type I.

```
> ADE1:= diff(y1(x), x) - y1(x) = 0:
> ADE2:= diff(y2(x),x,x)=6*y2(x)^2+x: #the transcendent
> NLDE:-composeDalg([ADE1,ADE2],[y1(x),y2(x)],z(x))
```

$$
\begin{aligned}
&24x\left(\frac{d}{dx}z(x)\right)^2 z(x)^4 + z(x)^6 - 2z(x)^5\left(\frac{d^3}{dx^3}z(x)\right) + 6\left(\frac{d^2}{dx^2}z(x)\right)\left(\frac{d}{dx}z(x)\right)z(x)^4 \\
&+\left(\frac{d^3}{dx^3}z(x)\right)^2 z(x)^4 - 4\left(\frac{d}{dx}z(x)\right)^3 z(x)^3 - 24\left(\frac{d^2}{dx^2}z(x)\right)\left(\frac{d}{dx}z(x)\right)^2 z(x)^3 \\
&-6\left(\frac{d^3}{dx^3}z(x)\right)\left(\frac{d^2}{dx^2}z(x)\right)\left(\frac{d}{dx}z(x)\right)z(x)^3 + 24\left(\frac{d}{dx}z(x)\right)^4 z(x)^2 \\
&+4\left(\frac{d^3}{dx^3}z(x)\right)\left(\frac{d}{dx}z(x)\right)^3 z(x)^2 + 9z(x)^2\left(\frac{d}{dx}z(x)\right)^2\left(\frac{d^2}{dx^2}z(x)\right)^2 \\
&-12z(x)\left(\frac{d}{dx}z(x)\right)^4\left(\frac{d^2}{dx^2}z(x)\right) + 4\left(\frac{d}{dx}z(x)\right)^6 = 0
\end{aligned}
\tag{18}
$$

The NLDE Package

Example (Functional Inverse)

Functional inverse of the Weierstrass elliptic function $\wp$.

```
> ADE:=diff(y(x),x)^2=4*y(x)^3-g2*y(x)-g3:
> invDalg(ADE,y(x),z(x))
```

$$1 + \left(-4x^3 + g2x + g3\right)\left(\frac{d}{dx}z(x)\right)^2 = 0. \tag{19}$$

[Ait El Manssour, Sattelberger, and T., 2023] and [T., 2023]

Take away. Note that:

- Every rational expression of D-algebraic functions satisfies a computable ADE of order at most the sum of the orders of the ADEs defining those D-algebraic functions.
- The composition of two D-algebraic functions is a D-algebraic function that fulfils a computable ADE whose order is at most the sum of the orders of the two defining ADEs.
- The derivative of a D-algebraic function is a D-algebraic function satisfying a computable ADE of the same order as the ADE defining the given function.
- The functional inverse of a D-algebraic function is a D-algebraic function that fulfils a computable ADE of the same order as the ADE defining the given function.

If $\wp(x)$ satisfies

$$y'(x)^2 = 4y(x)^3 - g_2\, y(x) - g_3.$$

Then the functional inverse $\wp^{-1}$ of $\wp$ satisfies

$$1 + \left(-4x^3 + g2x + g3\right) y'(x)^2 = 0.$$

See more about **NLDE** at **https://mathrepo.mis.mpg.de/OperationsForDAlgebraicFunctions/**

Thank You!