Subsequences and rational recursions for solutions of polynomial difference equations

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1 Introduction

Let r be a nonnegative integer and p an irreducible polynomial in $r+1$ variables over a field $\mathbb{K}$ of characteristic zero. Consider a sequence $(s(n))_{n\in\mathbb{N}}$ over $\mathbb{K}$ such that, for sufficiently large n,

$$q_{s,n}(x) := p(s(n), s(n{+}1), \dots, s(n{+}r{-}1), x) \in \mathbb{K}[x]\backslash\{0\} \quad \text{and} \quad q_{s,n}(s(n{+}r)) = 0.$$

We say that $(s(n))_{n\in\mathbb{N}}$ is *D-algebraic* with defining polynomial p. We here restrict our attention to this definition. We also assume that the order r is as small as possible. When $q_{s,n}$ is linear, we say that $(s(n))_n$ is a *rational recursive* sequence; this implies that for large n, the coefficient of x in $q_{s,n}$ is nonzero. The motivation for studying such sequences arises from theoretical computer science [2].

We discuss two theorems. The first concerns the closure property for subsequences of D-algebraic sequences indexed by arithmetic progressions.

Secondly, for a recurrence equation of the form

$$c_r(n)s(n+r) + \cdots + c_0(n)s(n) = 0, \quad c_r(n) \neq 0 \text{ for large } n, \tag{1}$$

where each coefficient $c_j(n)$ is D-algebraic with a linear p, we discuss an extension of the algorithm from [7] to show that the sequence $(s(n))_n$ is rational recursive. In [3], such a sequence $(s(n))_n$ is called *C^2-finite*. Note that the condition $c_r(n) \neq 0$ for large n is satisfied when $(c_r(n))_n$ is *non-degenerate* [1]. This means that its subsequences along arithmetic progressions do not satisfy lower-order linear polynomials.

This extended abstract is based on revisions of the preprint [5].

2 Subsequences

We prove closure properties for D-algebraic sequences by constructing dynamical systems that model the corresponding operations. As we know how to recover input-output equations from these systems, the closure properties follow.

For subsequences, we focus on gaps between consecutive terms. Let $(s(n))_n$ be D-algebraic of order r and defining polynomial p. Let d be a positive integer, r_0

and r_1, such that $r-1 = dr_1 + r_0, 0 \leq r_0 < d$. Consider $(t_j(n))_n := (s(dn+j))_n$, $0 \leq j < d$. Without loss of generality, we take $n = dk$. Then one can write

$$q_{s,n}(x) = p(t_0(k), t_1(k), \ldots, t_{r_0}(d(k+r_1)), x).$$

These are enough preliminaries to picture [5, Theorem 5.1].

Theorem 1. *Let d be a positive integer. If $(s(n))_n$ is D-algebraic of order r, then so is $(s(dn+j))_n, 0 \leq j < d$.*

Example 1 (Catalan numbers at $(3n)_{n\in\mathbb{N}}$). The sequence $(s(n))_n := (C_{3n})_n$, where $C_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number satisfies the recurrence equation

$$\begin{aligned}
&343597383680s(n)^3s(n+1)^3-69004689408s(n)^3s(n+1)^2s(n+2)+5s(n+1)^3s(n+2)^3\\
&-83243160s(n)^3s(n+2)^3-1258291200s(n)^2s(n+1)^4+266514432s(n)^2s(n+1)^3s(n+2)\\
&-26883000s(n)^2s(n+1)^2s(n+2)^2+1043658s(n)^2s(n+1)s(n+2)^3-122880s(n)s(n+1)^5\\
&-101544s(n)s(n+1)^4s(n+2)+65067s(n)s(n+1)^3s(n+2)^2-4113s(n)s(n+1)^2s(n+2)^3\\
&+1400s(n+1)^6-30s(n+1)^5s(n+2)-75s(n+1)^4s(n+2)^2+4274823168s(n)^3s(n+1)s(n+2)^2=0.
\end{aligned}$$

This equation is deduced from the rational recursive definition of $(C_n)_n$ given by

$$C_{n+2} = \frac{2C_{n+1}\left(8C_n + C_{n+1}\right)}{10C_n - C_{n+1}},\ C_0 = 1, C_1 = 1.$$

Using the method of guessing, the author was able to compute more second-order recurrence equations for $(C_{dn})_n$, $d \in \{1,2,3,4,5\}$ (see [6, Conjecture 5.2]).

3 Rational recursions

Recall that a C-finite sequence solves a linear recurrence equation with constant coefficients. A C^2-finite sequence is a solution to a linear recurrence equation with C-finite term coefficients [3]. The result from [7] deals with the case where the coefficients in (1) are polynomials. Such sequences are called D-finite (holonomic) or P-recursive and can be seen as D-algebraic over $\mathbb{K}(n)$.

We sketch the result for some first-order C^2-finite sequences and provide the general algorithm without its proof. We seek generic assumptions for the correctness of the algorithm.

Let $(s(n))_n$ be a C^2-finite sequence of *nonzero germ* such that

$$c_1(n)\,s(n+1) + c_0(n)\,s(n) = 0, \tag{2}$$

$$\begin{cases}\alpha_{1,1}\,c_1(n+1) + \alpha_{1,0}\,c_1(n) = c_1(n+2)\\ \alpha_{0,1}\,c_0(n+1) + \alpha_{0,0}\,c_0(n) = c_0(n+2)\end{cases}, \tag{3}$$

where $\alpha_{i,j} \in \mathbb{K}$, $i,j \in \{0,1\}$. Since $(s(n))_n$ is first-order with nonzero germ we must have $s(n) \neq 0$ for large n.

We eliminate the c_i's from the shifts of (2) by incremental substitution. We start by eliminating $c_1(n)$ and its shifts. From (2) and its first shift, we have

$$c_1(n) = -\frac{c_0(n)\, s(n)}{s(n+1)}, \qquad c_1(n+1) = -\frac{c_0(n+1)\, s(n+1)}{s(n+2)}. \tag{4}$$

We now proceed with the elimination of $c_0(n)$. We consider the second shift of (2) with the corresponding substitutions using (3) and (4). We obtain

$$c_0(n) = -\frac{s(n+1)\left(s(n+2)^2\, \alpha_{0,1} - s(n+1)\, s(n+3)\, \alpha_{1,1}\right) c_0(n+1)}{s(n+2)\,(s(n+1)\, s(n+2)\, \alpha_{0,0} - s(n)\, s(n+3)\, \alpha_{1,0})}. \tag{5}$$

The denominator provides one condition on three consecutive terms in the sequence. We call that $(\mathcal{C}_0(n))$. Observe that if $(\mathcal{C}_0(n))$ is not satisfied for all $n > N_0 \in \mathbb{N}$, then $(s(n))_{n \geq N_0}$ is rational recursive of order 3.
Since under $(\mathcal{C}_0(n))$, all shifts of C-finite coefficients can now be written as rational multiples of $c_0(n+1)$, it follows that their substitution in the third shift of (2) yields the product of $c_0(n+1)$ and a rational expression in the shifts of $s(n)$, in which $s(n+4)$ appears linearly. Hence, after canceling $c_0(n+1)$ we deduce the desired recursion:

$$s(n+4) = \frac{\mathsf{N}}{\mathsf{D}} := \frac{\mathsf{N}(s(n), \ldots, s(n+3))}{\mathsf{D}(s(n), \ldots, s(n+3))},$$

where

$$\begin{aligned}\mathsf{N} = -s(n+3)\,\big(&-s(n+2)\, s(n)\, s(n+3)\, \alpha_{0,1}^2 \alpha_{1,0} + s(n+1)^2\, s(n+3)\, \alpha_{0,0}\alpha_{0,1}\alpha_{1,1} \\ &+ s(n+2)^2\, s(n+1)\, \alpha_{0,0}^2 - s(n+2)\, s(n)\, s(n+3)\, \alpha_{0,0}\alpha_{1,0}\big)\,,\end{aligned} \tag{6}$$

$$\begin{aligned}\mathsf{D} = {}& s(n+2)^2\, s(n)\, \alpha_{0,1}\alpha_{1,0}\alpha_{1,1} - s(n+2)\, s(n+1)^2\, \alpha_{0,0}\alpha_{1,1}^2 \\ &- s(n+2)\, s(n+1)^2\, \alpha_{0,0}\alpha_{1,0} + s(n+1)\, s(n)\, s(n+3)\, \alpha_{1,0}^2.\end{aligned} \tag{7}$$

We denote by $(\mathcal{C}_1(n))$ the condition on the denominator D. Again, if this condition is not continuously met for large n, then D provides a recursion for $(s(n))_n$. In summary, we have the following proposition adapted from [5, Prop. 4.8].

Proposition 1. *Let $(s(n))_n$ be a first-order C^2-finite sequence of nonzero germ with second-order C-finite term coefficients. For all n fulfilling $(\mathcal{C}_0(n))$ and $(\mathcal{C}_1(n))$, the terms of $(s(n))_n$ satisfy a rational recursion of order at most 5.*

Note that if some of the coefficients in the C^2-finite equation are nonzero constants, then in the third shift of that equation, not all C-finite coefficients (including constants) would be expressible as a rational function in the shifts of $s(n)$ multiplied by some $c_i(n+1)$, $i \in \{0, 1\}$. Therefore, one more shift would be required to find the rational recursion. This is why the bound is 5 and not 4.
The above elimination process can be turned into an algorithm for C^2-finite sequences of arbitrary order. We denote by σ the shift operator.

Algorithm 1 C^2-finite to D-algebraic rational recursion

```
Input:
  – $C^2$-finite equation $(p) : c_0(n)s(n) + \cdots + c_l(n)s(n+l) = 0,\ l > 0$.
  – $C$-finite equations $(q_j) : c_j(n+r_j) = \alpha_{i,0}c_j(n) + \cdots + \alpha_{j,r_j-1}c(n+r_j-1)$,
    $j \in \{0, \ldots, l\}$.
Output: A rational recursion of order at most $l + \sum_{j=0}^{l} r_j$.
  1. Set $r_l := r_l - 1$, unless some $r_j = 0$ and $c_j(n) \neq 0$.
  2. For each $j$, $0 \leq j \leq l$ do:
    2.1. For each $k$, $0 \leq k \leq r_j - 1$, do:
      2.1.1. $c_j(n+k) := \text{solve}((p), c_j(n+k))$. //root of a univariate linear polynomial.
      2.1.2. $(p) := \sigma((p))$;
      2.1.3. Update $(p)$ and the $(q_i)$, $0 \leq i \leq l$, by substituting $c_j(n+k)$ as solved.
  3. Let $r := l + \sum_{j=0}^{l} r_j$.
  4. Return "$s(n+r) = \text{solve}((p), s(n+r))$" or its associated difference polynomial.
```

The elimination procedure in Algorithm 1 is inherently similar to the linear algebra computations detailed in [4, Section 4]. Specifically, Theorem 28 reveals that this method invariably yields a rational recursion when applied to C^2-finite equations. Jimenez-Pastor later confirmed this property to us (private communication). However, to ensure correctness, we must investigate conditions to avoid zero divisors.

Our implementation of Algorithm 1 is available in the `NLDE` package via the command

`DalgSeq:-CCfiniteToSimpleRatrec`,

from the subpackage `DalgSeq`. This command is much more efficient than our other implementation `DalgSeq:-CCfiniteToDalg` that is based on Gröbner bases elimination. The Maple worksheet `C2-finite-to-ratrec.mw` (also printed in PDF format) from the `NLDE` GitHub page provides an example with the syntax.

References

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